

Disk relations for tree amplitudes in minimal coupling theory of gauge field and gravity

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Abstract

KLT relations on S_2 factorize closed string amplitudes into product of open string tree amplitudes. The field theory limits of KLT factorization relations hold in minimal coupling theory of gauge field and gravity. In this paper, we consider the field theory limits of relations on D_2 . Though the relations on D_2 and KLT factorization relations hold on worldsheets with different topologies, we find the field theory limits of D_2 relations also hold in minimal coupling theory of gauge field and gravity. We use the D_2 relations to give three- and four-point tree amplitudes where gluons are minimally coupled to gravitons. We also give a discussion on general tree amplitudes for minimal coupling of gauge field and gravity. In general, any tree amplitude with M gravitons in addition to N gluons can be given by pure-gluon tree amplitudes with $N + 2M$ legs.

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I. INTRODUCTION

Superstring theories are theories containing both gravitational and gauge interactions[1, 2]. They offer a possible way to unify the gauge field and gravity. In string theory, gravitons and gauge particles can correspond to massless states of closed and open strings. Then the relations between closed and open strings imply the relations between gravity and gauge field.

There are many relationships between gauge field and gravity in string theory such as AdS/CFT[3] correspondence and the perturbative relations for amplitudes on sphere(S_2)[4], disk(D_2)[5–10] and real projective plane(RP_2)[5]. The perturbative relations in S_2 case named KLT relation[4] factorize the amplitudes on S_2 into products of two amplitudes for open strings corresponding to the left- and the right-moving sectors of closed strings. However, the amplitudes on D_2 cannot be factorized into two sectors, because the boundary of D_2 connect the two sectors into a single one. We should use the new relations instead of KLT factorization relations on D_2 . KLT factorization relations and D_2 relations hold on worldsheets with different topologies.

In field theory limit, KLT factorization relations allow one to obtain gravity amplitudes from gauge theory ones[11–16]. They can also be used in many theories of gravity-matter couplings[17, 18]. An important application is that the low energy limits of KLT relations can be used to calculate the tree amplitudes for gluons minimally coupled to gravitons. In this case, using the relations, the amplitudes with gluons and gravitons can be factorized into products of amplitudes in left- and right-moving sectors. Amplitudes in one sector are pure gauge partial amplitudes while those in the other sector are partial amplitudes with gluons and scalars. These relations for gauge-gravity minimal coupling are based on the

structure of heterotic strings[1, 2, 19–21]. In fact, in heterotic string theories, gauge degrees of freedom are taken by Lorentz singlets in one sector of closed strings.

There is another way to incorporate gauge degrees of freedom into string theory. In theories containing open strings such as Type I theory, one can add Chan-Paton factors[1, 2, 22, 23] to the ends of open strings. The interactions between closed and open strings at tree-level are on D_2 . In the field theory limits of this case, the tree amplitudes for gauge-gravity interaction and those for pure gauge field are connected via D_2 relations. Then a question arises: In what kind of theory for gauge-gravity interactions do the field limits of D_2 relations hold? Or do the field theory limits of D_2 relations also hold in minimal coupling theory of gauge and gravity?

In this paper, we study the amplitudes in minimal coupling theory of gauge field and gravity. We find though the KLT factorization relations and the D_2 relations hold on worldsheets with different topologies in string theory, the field theory limits of D_2 relations also hold in minimal coupling theory of gauge field and gravity. D_2 relations in minimal coupling theory of gauge field and gravity are based on the disk structure. They give a new understanding on gauge-gravity interaction.

The structure of this paper is as follows. In Section II, we give an introduction to KLT relations and D_2 relations. We also give the low energy limits of D_2 relations for amplitudes with three and four legs. In Section III, we use the low energy limits of D_2 relations to give the tree amplitudes for gauge-gravity interaction. We find the amplitudes given by D_2 relations are same with those given by KLT factorization relations. Then the D_2 relations give the amplitudes in minimal coupling theory of gauge field and gravity. In Section IV, we consider the D_2 relations for general tree amplitudes where gluons are minimally coupled to gravitons. We first study the mixed amplitudes where all the legs take positive helicity or

only one leg take negative helicity. In these cases, the D_2 relations hold trivially. Then we study the tree amplitudes with one and two gravitons in addition to N gluons where $N - 2$ gluons as well as all the gravitons take positive helicity and two gluons take negative helicity. We will show the D_2 relations also hold in this case. The discussions can be extended to tree amplitudes where N gluons minimally coupled to M gravitons with arbitrary helicity configurations. In general, any tree amplitudes for gauge-gravity minimal coupling can be expressed by partial tree amplitudes with $N + 2M$ gluons via D_2 relations. Our conclusions are given in Section V. Some useful properties of spinor helicity formalism are given in Appendix A.

II. KLT RELATIONS VERSUS D_2 RELATIONS

KLT relations[4] in string theory are the relations between amplitudes for closed strings on S_2 and open string tree amplitudes. KLT relations factorize amplitudes for closed strings on S_2 into products of two open string tree amplitudes corresponding to the left- and right-moving sectors except for a phase factor¹

$$\mathcal{M}_{S_2}^{(N)} \sim \kappa^{N-2} \sum_{P, P'} \mathcal{A}^{(N)}(P) \bar{\mathcal{A}}^{(N)}(P') e^{i\pi F(P, P')}, \quad (1)$$

Where $\mathcal{M}_{S_2}^{(N)}$ is N -point amplitude on S_2 while $\mathcal{A}^{(N)}$ and $\bar{\mathcal{A}}^{(N)}$ are the partial tree amplitudes for open strings corresponding to the left- and right-moving sectors. The phase factor only depends on the permutations P and P' of the legs in left- and right-moving sectors. The terms in KLT relations can be reduced by contour deformations. In the reduced form, the phase factors become sine functions. After taking the field theory limit $\alpha' \rightarrow 0$, the KLT

¹ In this paper, we use \sim to omit a proportional factor which does not affect our discussion.

relations for three- and four-point amplitudes are given as

$$\mathcal{M}(1, 2, 3) \sim \kappa \mathcal{A}(1, 2, 3) \bar{\mathcal{A}}(1, 2, 3), \quad (2a)$$

$$\mathcal{M}(1, 2, 3, 4) \sim \kappa^2 (-i) s_{12} \mathcal{A}(1, 2, 3, 4) \bar{\mathcal{A}}(1, 2, 4, 3). \quad (2b)$$

In field theory limits, KLT relations factorize the pure-graviton tree amplitudes into products of tree amplitudes for gluons corresponding to left- and right-moving sectors. The tree amplitudes where gravitons are minimally coupled to gluons can also be factorized by KLT relations. In this situation, the two sectors of a graviton with helicity ± 2 correspond to two gluons with helicity ± 1 , while the two sectors of a gluon with helicity ± 1 correspond to one gluon with helicity ± 1 and one scalar particle. The gauge degrees of freedom are taken by the scalar field in one sector. With these correspondences, KLT relations express the amplitudes for N gluons and M gravitons by products of amplitudes in left- and right-moving sectors. The amplitudes in the left-moving sector are pure-gluon partial tree amplitudes with $N+M$ legs while the amplitudes in the right-moving sector are partial tree amplitudes for M gluons and N scalar particles. Using the Feynman rules given in [17], one can calculate the partial amplitudes in the left- and right-moving sectors, then the amplitudes where gluons are minimally coupled with gravitons can be given by KLT relations.

The KLT factorization relations in string theory hold on S_2 . But they do not hold on D_2 . In D_2 case, the left- and right-moving sectors of closed strings are connected into a single one[5]. The relations between amplitudes for N open strings in addition to M closed strings on D_2 and open string tree amplitudes are given as

$$\mathcal{M}_{D_2}^{(N,M)} \sim g^{N-2} \kappa^M \sum_P \mathcal{A}^{(N,2M)}(P) e^{i\pi\Theta'(P)}. \quad (3)$$

In the relation (3), we do not introduce the Chan-Paton degrees of freedom. On D_2 , one can add Chan-Paton factor to the ends of open strings to incorporate gauge degree of freedom.

Then the amplitudes on D_2 can be given by color decomposed form

$$\begin{aligned} & \mathcal{M}(1_o^{a_1}, \dots, N_o^{a_N}, (N+1)_c, \dots, (N+M)_c) \\ &= \sum_{\sigma} \text{Tr} \left(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(N)}} \right) \mathcal{A}(\sigma(1_o), \dots, \sigma(N_o), (N+1)_c, \dots, (N+M)_c), \end{aligned} \quad (4)$$

where $i_o^a (i = 1, \dots, N)$ denote open string legs with Chan-Paton degrees of freedoms and $j_c (j = N+1, \dots, N+M)$ denote closed string legs. σ runs over the set of non cyclic permutations of the open strings. Then D_2 relations give the partial amplitudes $\mathcal{A}(\sigma(1_o), \dots, \sigma(N_o), (N+1)_c, \dots, (N+M)_c)$ for a given permutation of open strings by pure open string amplitudes:

$$\mathcal{A}(\sigma(1_o), \dots, \sigma(N_o), (N+1)_c, \dots, (N+M)_c) = \sum_P e^{i\pi\Theta'(P'')} \mathcal{A}^{(N,2M)}(P''). \quad (5)$$

where P'' are all the permutations of the $N+2M$ external legs which preserve the relative positions of the open strings $1_o, \dots, N_o$. This expression implies that for a given permutation of the open strings on the boundary of D_2 , any closed string can split into two open strings inserted on the boundary of D_2 . Using contour deformations, the relations (5) can also be reduced[6]. In the reduced form of the relations, the phase factors become sine functions.

In field theory limits, D_2 relations give tree amplitudes with N gluons and M gravitons by pure-gluon amplitudes with $N+2M$ legs. They are different from the KLT factorization relations which factorize amplitudes with $N+M$ legs into products of two amplitudes with $N+M$ legs. For $M=0$, D_2 relations trivially give the pure-gluon amplitudes. Then we do not need to consider $M=0$ case. Since the generators of the gauge group satisfy $\text{Tr}(T^a) = 0$, we also do not need to consider $M=1$ case. After taking the field theory limit $\alpha' \rightarrow 0$, the D_2 relations for two-gluon one-graviton tree amplitude $\mathcal{A}(1_g, 2_g, 3_h)$, three-gluon one-graviton tree amplitude $\mathcal{A}(1_g, 2_g, 3_g, 4_h)$ and two-gluon two-graviton tree amplitude $\mathcal{A}(1_g, 2_g, 3_h, 4_h)$

are given as²

$$\mathcal{A}(1_g, 2_g, 3_h) \sim \kappa s_{12} \mathcal{A}(1_g, 2_g, 3_g, 4_g), \quad (6a)$$

$$\mathcal{A}(1_g, 2_g, 3_g, 4_h) \sim g \kappa s_{13} \mathcal{A}(1_g, 5_g, 2_g, 4_g, 3_g), \quad (6b)$$

$$\mathcal{A}(1_g, 2_g, 3_h, 4_h) \sim \kappa^2 [s_{12}^2 \mathcal{A}(1_g, 6_g, 3_g, 5_g, 4_g, 2_g) - s_{12} s_{13} \mathcal{A}(1_g, 3_g, 5_g, 4_g, 2_g, 6_g)], \quad (6c)$$

where $s_{ij} = 2k_i \cdot k_j$, 3_g and 4_g have momentum $\frac{1}{2}k_3$ while 5_g and 6_g have momentum $\frac{1}{2}k_4$.

The total amplitude is derived by substitute the relations (6) into Eq. (4).

So far, we have seen, though KLT and D_2 relations hold on worldsheets with different topologies in string theory. They can both give the amplitudes for gauge-gravity coupling. KLT relations can given the amplitudes for gauge-gravity minimal coupling. Then we should consider the question: Can the field theory limits of the two different relations in string theory hold in a same theory for gauge-gravity coupling? In the next two sections, we will show the D_2 relations also hold in minimal coupling theory of gauge field and gravity.

III. D_2 RELATIONS FOR THREE- AND FOUR-POINT TREE AMPLITUDES

In this section, we use the D_2 relations (6) to give the three- and four-point tree amplitudes for gauge-gravity coupling. These results are same with those given by using KLT relations[17]. Thus D_2 relations also hold in minimal coupling theory of gauge field and gravity.

² In this paper, we use i_g and j_h to denote gluons and gravitons respectively. We do not consider the amplitudes where all the external legs are gravitons. We also do not consider the amplitudes where graviton exchanges between gluons, which contribute to higher order process.

A. Three-point tree amplitude

The only nontrivial three-point partial amplitude needed is $\mathcal{A}(1_g, 2_g, 3_h)$. To give this amplitude, we need to calculate the amplitudes $\mathcal{A}(1_g, 2_g, 3_g, 4_g)$ in which 3_g and 4_g correspond to the left- and right-moving sectors of the graviton 3_h . Using the color-ordered Feynman rules in [24] and replacing the momenta k_3 and k_4 by $\frac{1}{2}k_3$, we can get the amplitude $\mathcal{A}(1_g, 2_g, 3_g, 4_g)$. After substituting it into the relation (6a), the three-point amplitude $\mathcal{A}(1_g, 2_g, 3_h)$ is given

$$\mathcal{A}(1_g, 2_g, 3_h) \sim 2[-\epsilon_1 \cdot \epsilon_2 \epsilon_3^{\rho\sigma} k_{1\rho} k_{2\sigma} + \epsilon_1 \cdot k_2 \epsilon_{3\rho\sigma} k_{1\rho} \epsilon_{2\sigma} + \epsilon_2 \cdot k_1 \epsilon_3^{\rho\sigma} \epsilon_{1\rho} k_{2\sigma}], \quad (7)$$

where the physical conditions $\epsilon_1 \cdot k_1 = \epsilon_2 \cdot k_2 = 0$, $\epsilon_{3\rho\sigma} k_3^\rho = \epsilon_{3\rho\sigma} k_3^\sigma = 0$, momentum conservation $k_1^\mu + k_2^\mu + k_3^\mu = 0$ and the traceless condition of the polarization tensor of graviton $\epsilon_{3\rho}^\rho = 0$ have been used. This is same with that given by KLT relation(2a). Thus, D_2 relation can give the three-point tree amplitude where gluons are minimally coupled to graviton.

B. Four-point tree amplitudes

In this subsection, we study the four-point amplitudes $\mathcal{A}(1_g, 2_g, 3_g, 4_h)$ and $\mathcal{A}(1_g, 2_g, 3_h, 4_h)$. We use the spinor helicity formalism[24–26] to consider the four-point amplitudes. Useful properties of spinor helicity formalism are listed in Appendix A.

The two gluons corresponding to a graviton with helicity ± 2 take helicity ± 1 . Then the four-point tree amplitudes with all legs of positive helicity can be given by pure-gluon amplitudes with all legs of positive helicity via the relations (6b) and (6c). Because the pure-gluon partial amplitudes in which all gluons take the same helicity vanish, we have

$$\mathcal{A}(1_g^+, 2_g^+, 3_g^+, 4_h^+) = \mathcal{A}(1_g^+, 2_g^+, 3_h^+, 4_h^+) = 0. \quad (8)$$

The tree amplitudes with one gluon of negative helicity and other legs of positive helicity can be given by pure-gluon tree amplitudes where only one leg take negative helicity and other legs take positive helicity. Since the pure-gluon tree amplitudes with one leg of negative helicity and other legs of positive helicity vanish, we have

$$\mathcal{A}(1_g^-, 2_g^+, 3_g^+, 4_h^+) = \mathcal{A}(1_g^-, 2_g^+, 3_h^+, 4_h^+) = 0. \quad (9)$$

The tree amplitudes with one graviton of negative helicity and other legs of positive helicity can be given by pure-gluon MHV[27] amplitudes. In this pure-gluon MHV amplitude, the two gluons corresponding to the negative helicity graviton take negative helicity. Using the relation (6b) and the expression of MHV amplitude for gluons (A11), the amplitude $\mathcal{A}(1_g^+, 2_g^+, 3_g^+, 4_h^-)$ is given

$$\mathcal{A}(1_g^+, 2_g^+, 3_g^+, 4_h^-) \sim g^2 \kappa \mathcal{A}(1_g^+, 5_g^-, 2_g^+, 4_g^-, 3_g^+) = g \kappa s_{13} i \frac{\langle 54 \rangle^4}{\langle 15 \rangle \langle 52 \rangle \langle 24 \rangle \langle 43 \rangle \langle 31 \rangle} = 0, \quad (10)$$

where we have use $k_5 = k_4$. Similarly, $\mathcal{A}(1_g^+, 2_g^+, 3_h^-, 4_h^+) = 0$.

Now we consider the MHV amplitudes where two legs take negative helicity and others take positive helicity. There are three independent amplitudes $\mathcal{A}(1_g^-, 2_g^-, 3_g^+, 4_h^+)$, $\mathcal{A}(1_g^-, 2_g^-, 3_h^+, 4_h^+)$ and $\mathcal{A}(1_g^-, 2_g^+, 3_h^-, 4_h^+)$. With the D_2 relations (6b) and (6c), the first two amplitudes can be expressed by five- and six-point pure-gluon MHV amplitudes respectively. After using some properties of the spinor helicity formalism and the fact that the two gluons corresponding to one graviton take half of the momentum of the graviton, we get the first two amplitudes

$$\begin{aligned} \mathcal{A}(1_g^-, 2_g^-, 3_g^+, 4_h^+) &\sim g \kappa s_{13} \mathcal{A}(1_g^-, 5_g^+, 2_g^-, 4_g^+, 3_g^+) \\ &= g \kappa s_{13} i \frac{\langle 12 \rangle^4}{\langle 15 \rangle \langle 52 \rangle \langle 24 \rangle \langle 43 \rangle \langle 31 \rangle} \\ &\sim g \kappa \sqrt{2} \frac{\langle 12 \rangle^4 [12]}{\langle 14 \rangle \langle 24 \rangle \langle 34 \rangle^2}. \end{aligned} \quad (11)$$

$$\begin{aligned}
\mathcal{A}(1_g^-, 2_g^-, 3_h^+, 4_h^+) &\sim \kappa^2 [s_{12}^2 \mathcal{A}(1_g^-, 6_g^+, 3_g^+, 5_g^+, 4_g^+, 2_g^-) - s_{12}s_{13} \mathcal{A}(1_g^-, 3_g^+, 5_g^+, 4_g^+, 2_g^-, 6_g^+)] \\
&= \kappa^2 \left[s_{12}^2 i \frac{\langle 12 \rangle^4}{\langle 16 \rangle \langle 63 \rangle \langle 35 \rangle \langle 54 \rangle \langle 42 \rangle \langle 21 \rangle} - s_{12}s_{13} \frac{\langle 12 \rangle^4}{\langle 13 \rangle \langle 35 \rangle \langle 54 \rangle \langle 42 \rangle \langle 26 \rangle \langle 61 \rangle} \right] \\
&= 0.
\end{aligned} \tag{12}$$

To calculate $\mathcal{A}(1_g^-, 2_g^+, 3_h^-, 4_h^+)$, we need six-point non-MHV tree amplitudes for gluons $\mathcal{A}(1_g^-, 6_g^+, 3_g^-, 5_g^+, 4_g^-, 2_g^+)$ and $\mathcal{A}(1_g^-, 3_g^-, 5_g^+, 4_g^-, 2_g^+, 6_g^+)$. Using the tree amplitude with six gluons given in [25] and substituting k_3, k_4 and k_5, k_6 by $\frac{k_3}{2}$ and $\frac{k_4}{2}$ correspondingly, we get

$$\begin{aligned}
\mathcal{A}(1_g^-, 6_g^+, 3_g^-, 5_g^+, 4_g^-, 2_g^+) &= -16i \frac{[24]^4 \langle 13 \rangle^2 \langle 23 \rangle^2}{s_{12}^3 s_{23}^2}, \\
\mathcal{A}(1_g^-, 3_g^-, 5_g^+, 4_g^-, 2_g^+, 6_g^+) &= 16i \frac{[24]^4 \langle 13 \rangle^2 \langle 23 \rangle^2}{s_{12} s_{13}^2 s_{23}^2}.
\end{aligned} \tag{13}$$

Then we substitute the amplitudes (13) into the relation (6c). The amplitude $\mathcal{A}(1_g^-, 2_g^+, 3_h^-, 4_h^+)$ is given

$$\begin{aligned}
\mathcal{A}(1_g^-, 2_g^+, 3_h^-, 4_h^+) &\sim \kappa^2 [s_{12}^2 \mathcal{A}(1_g^-, 6_g^+, 3_g^-, 5_g^+, 4_g^-, 2_g^+) - s_{12}s_{13} \mathcal{A}(1_g^-, 3_g^-, 5_g^+, 4_g^-, 2_g^+, 6_g^+)] \\
&= -16i \frac{[24]^4 \langle 13 \rangle^2 \langle 23 \rangle^2}{s_{12} s_{23}^2} - 16i \frac{[24]^4 \langle 13 \rangle^2 \langle 23 \rangle^2}{s_{12} s_{13} s_{23}^2} \\
&\sim \frac{[24]^4 \langle 23 \rangle^2 \langle 13 \rangle^2}{s_{12} s_{23} s_{13}}.
\end{aligned} \tag{14}$$

So far, we have given all the independent three- and four-point amplitudes, other three- and four-point amplitudes can be derived from these amplitudes by using a parity transformation or performing an appropriate replacement on the external legs. These results are same with those given by KLT relations [17]. Then the three- and four-point tree amplitudes for gluons minimally coupled to graviton satisfy D_2 relations. We then expect the D_2 relations hold in all amplitudes where gluons are minimally coupled to gravitons.

IV. GENERAL DISCUSSIONS ON D_2 RELATIONS FOR TREE AMPLITUDES IN MINIMAL COUPLING THEORY OF GAUGE FIELD AND GRAVITY

In this section, we study general tree amplitudes in minimal coupling theory of gauge field and gravity. D_2 relations can give amplitudes with N gluons and M gravitons by sum of pure-gluon partial amplitudes with $N + 2M$ legs except for appropriate factors. If all the gluons and gravitons take positive helicity, the amplitude must vanish. This is because the pure-gluon amplitudes with all legs of positive helicity vanish.

The tree amplitudes with one gluon of negative helicity and other legs of positive helicity can be expressed as sum of pure-gluon amplitudes with one leg of negative helicity and other legs of positive helicity. Then these amplitudes also vanish. The amplitudes with one graviton of negative helicity and other $N + M - 1$ legs of positive helicity are expressed by sum of MHV $(N + 2M)$ -gluon tree amplitudes where the two gluons corresponding to the negative helicity graviton take negative helicity. The two negative helicity gluons in the $(N + 2M)$ -gluon amplitudes take the same momentum. Considering the antisymmetry of the spinor products (A6), these MHV tree amplitudes for $N + 2M$ gluons vanish. Thus the amplitudes with one graviton of negative helicity and other $N + M - 1$ legs of positive helicity must vanish.

The results above given by D_2 relations are same with those given by KLT relations. Then the D_2 relations can give the amplitudes with all the legs of positive helicity for gauge-graviton minimal coupling. They also give amplitudes with one leg of negative helicity and other legs of positive helicity for gauge-gravity minimal coupling.

Though the trivial cases are easy to consider, D_2 relations for nontrivial helicity configurations are not so clear. In the Subsections IV A and IV B, we will give discussions

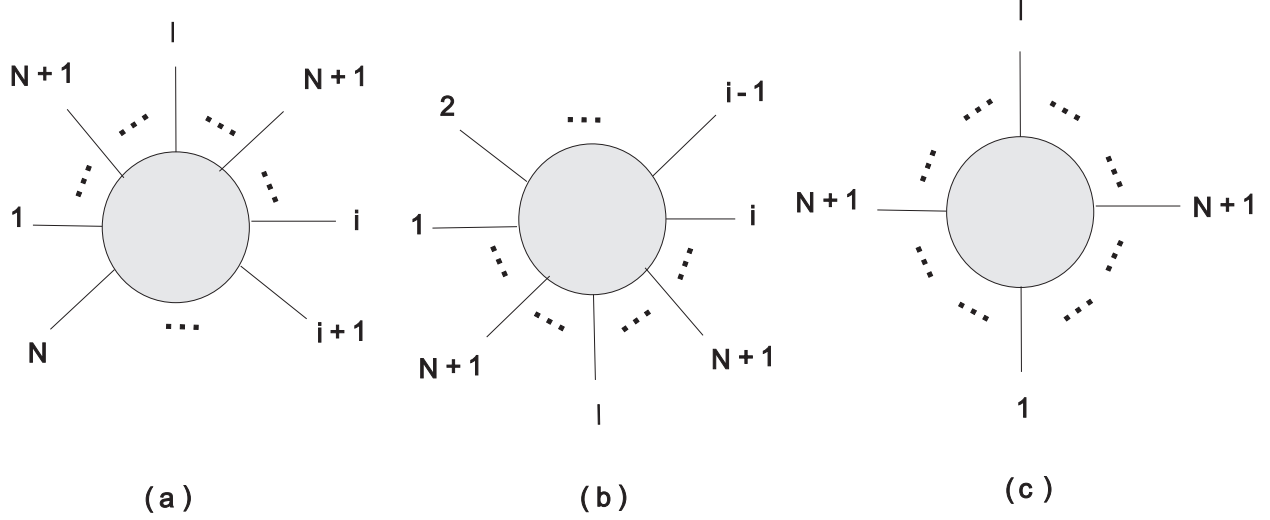


FIG. 1: Positions of the two gluons corresponding to the graviton for (a) $1 < l < i$, (b) $i < l \leq N$ and (c) the expression independent of helicity configuration.

on amplitudes with one and two gravitons in addition to N gluons. Here two gluons take negative helicity and other legs take positive helicity. In Subsection IV C, we extend the discussions to arbitrary tree amplitudes where N gluons are minimally coupled to M gravitons. Two gluons take negative helicity and the other $N + M - 2$ legs take positive helicity. We then extend the D_2 relations for amplitudes with one graviton in addition to N gluons to relations independent of the helicity configurations. We suggest D_2 relations should hold for any helicity configurations. The D_2 relation for a given amplitude is not unique. Different relations can be related by relations among partial amplitudes of gauge field.

A. D_2 relations for tree amplitudes with one positive helicity graviton, $N - 2$

positive helicity gluons and two negative helicity gluons

The tree amplitudes with two gluons of negative helicity and other legs of positive helicity can be expressed by [28, 29]

$$\begin{aligned} & \mathcal{A}(1_g^-, 2_g^+, \dots, i_g^-, \dots, N_g^+, (N+1)_h^+, \dots, (N+M)_h^+) \\ &= ig^{N-2} \left(-\frac{\kappa}{2}\right)^M \frac{\langle 1i \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle N1 \rangle} S(1, i, \{h^+\}, \{g^+\}), \end{aligned} \quad (15)$$

where

$$\begin{aligned} S(i, j, \{h^+\}, \{g^+\}) &= \left(\prod_{m \in \{h^+\}} \frac{d}{da_m} \right) \\ &\times \prod_{l \in \{g^+\}} \exp \left[\sum_{n_1 \in \{h^+\}} a_{n_1} \frac{\langle li \rangle \langle lj \rangle [ln_1]}{\langle n_1 i \rangle \langle n_1 j \rangle \langle ln_1 \rangle} \times \exp \left[\sum_{n_2 \in \{h^+\}, n_2 \neq n_1} a_{n_2} \frac{\langle n_1 i \rangle \langle n_1 j \rangle [n_1 n_2]}{\langle n_2 i \rangle \langle n_2 j \rangle \langle n_1 n_2 \rangle} \exp [\dots] \right] \right] \Big|_{a_j=0}. \end{aligned} \quad (16)$$

In $M = 1$ case where all the legs take positive helicity except 1 and i , the amplitude is reduced to

$$\begin{aligned} & \mathcal{A}(1_g^-, 2_g^+, \dots, i_g^-, \dots, N_g^+, (N+1)_h^+) \\ &= ig^{N-2} \left(-\frac{\kappa}{2}\right) \frac{\langle 1i \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle N1 \rangle} \sum_{l \in \{g^+\}} \frac{\langle l1 \rangle \langle li \rangle [l, N+1]}{\langle N+1, 1 \rangle \langle N+1, i \rangle \langle l, N+1 \rangle} \\ &= ig^{N-2} \left(\frac{\kappa}{2}\right) \sum_{l \in \{g^+\}} \frac{\langle 1i \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle N1 \rangle} \langle l, N+1 \rangle [l, N+1] \\ &\times \frac{\langle 1l \rangle}{\langle 1, N+1 \rangle \langle N+1, l \rangle} \frac{\langle li \rangle}{\langle l, N+1 \rangle \langle N+1, i \rangle}. \end{aligned} \quad (17)$$

In the equation above, $\frac{\langle 1i \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle N1 \rangle}$ is just the MHV amplitude for N gluons with the permutation $1, \dots, N$. Using (A5), we have $\langle l, N+1 \rangle [l, N+1] = s_{l, N+1}$. With the eikonal identity (A9), $\frac{\langle 1l \rangle}{\langle 1, N+1 \rangle \langle N+1, l \rangle}$ and $\frac{\langle li \rangle}{\langle l, N+1 \rangle \langle N+1, i \rangle}$ can split into sums of terms over all the legs

between 1, l and l , i respectively. For $1 < l < i$,

$$\begin{aligned}\frac{\langle 1l \rangle}{\langle 1, N+1 \rangle \langle N+1, l \rangle} &= \sum_{r=1}^{l-1} \frac{\langle r, r+1 \rangle}{\langle r, N+1 \rangle \langle N+1, r+1 \rangle}, \\ \frac{\langle li \rangle}{\langle l, N+1 \rangle \langle N+1, i \rangle} &= \sum_{t=l}^{i-1} \frac{\langle t, t+1 \rangle}{\langle t, N+1 \rangle \langle N+1, t+1 \rangle}.\end{aligned}\tag{18}$$

For $i < l \leq N$,

$$\begin{aligned}\frac{\langle il \rangle}{\langle i, N+1 \rangle \langle N+1, l \rangle} &= \sum_{r=i}^{l-1} \frac{\langle r, r+1 \rangle}{\langle r, N+1 \rangle \langle N+1, r+1 \rangle}, \\ \frac{\langle l1 \rangle}{\langle l, N+1 \rangle \langle N+1, 1 \rangle} &= \sum_{t=l}^N \frac{\langle t, t+1 \rangle}{\langle t, N+1 \rangle \langle N+1, t+1 \rangle},\end{aligned}\tag{19}$$

where we define $t+1 = 1$ for $t = N$. The amplitude then becomes

$$\begin{aligned}\mathcal{A}(1_g^-, 2_g^+, \dots, i_g^-, \dots, N_g^+, (N+1)_h^+) \\ = ig^{N-2} \left(\frac{\kappa}{2} \right) \left(\sum_{1 < l < i} s_{l, N+1} \sum_{r=1}^{l-1} \sum_{t=l}^{i-1} + \sum_{i < l \leq N} s_{l, N+1} \sum_{r=i}^{l-1} \sum_{t=l}^N \right) \\ \cdot \frac{\langle 1i \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle N1 \rangle} \frac{\langle r, r+1 \rangle}{\langle r, N+1 \rangle \langle N+1, r+1 \rangle} \frac{\langle t, t+1 \rangle}{\langle t, N+1 \rangle \langle N+1, t+1 \rangle},\end{aligned}\tag{20}$$

where for $t = N$ in the sum over t , we define $k_{t+1} \equiv k_1$. For a given r , the numerator of $\frac{\langle r, r+1 \rangle}{\langle r, N+1 \rangle \langle N+1, r+1 \rangle}$ is a spinor product for two adjacent points. Then we can remove $\langle r, r+1 \rangle$ in $\frac{\langle r, r+1 \rangle}{\langle r, N+1 \rangle \langle N+1, r+1 \rangle}$ and $\frac{\langle 1i \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle N1 \rangle}$ simultaneously. Then $\langle r, r+1 \rangle$ in the denominator of $\frac{\langle 1i \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle N1 \rangle}$ is replaced by $\langle r, N+1 \rangle \langle N+1, r+1 \rangle$. After this replacement, $\frac{\langle 1i \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle N1 \rangle}$ becomes $\frac{\langle 1i \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle r, N+1 \rangle \langle N+1, r+1 \rangle \dots \langle N1 \rangle}$ which is the $(N+1)$ -gluon MHV tree amplitude with the permutation $1, 2, \dots, r, N+1, r+1, \dots, N$. Thus $\frac{\langle r, r+1 \rangle}{\langle r, N+1 \rangle \langle N+1, r+1 \rangle}$ just insert a gluon with momentum k_{N+1} between the two gluons r and $r+1$. For a given l ($1 < l < i$), the sum over r for $1 \leq r \leq l-1$ becomes sum over all the possible insertions between 1 and l . For $i < l \leq N$, the sum over r for $i \leq r \leq l-1$ becomes sum over all the possible insertions between i and l . After a similar discussion, $\frac{\langle t, t+1 \rangle}{\langle t, N+1 \rangle \langle N+1, t+1 \rangle}$ insert a gluon with momentum k_{N+1} at the positions between l , i for $1 < l < i$ and between i , 1 for $i < l \leq N$. The two

gluons with momentum k_{N+1} just correspond to the left- and right-moving sectors of the graviton³. Then $\frac{\langle 1i \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle N1 \rangle}$ becomes $\frac{\langle 1i \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle r, N+1 \rangle \langle N+1, r+1 \rangle \dots \langle t, N+1 \rangle \langle N+1, t \rangle \dots \langle N1 \rangle}$ which is the MHV tree amplitude with $N + 2$ gluons.

Thus, the amplitude can be considered as a sum of terms. In each term, there is an MHV tree amplitude for $N + 2$ gluons. Two of the $N + 2$ gluons take the momentum k_{N+1} . Then the amplitude satisfies the relation

$$\begin{aligned} & \mathcal{A}(1_g^-, 2_g^+, \dots, i_g^-, \dots, N_g^+, (N+1)_h^+) \\ &= i g^{N-2} \left(\frac{\kappa}{2} \right) \sum_{l \in \{g^+\}} s_{l, N+1} \sum_P \mathcal{A}_{MHV}^{N+2}(P), \end{aligned} \quad (21)$$

Where for any given l in $\{g^+\}$, we sum over permutations P . P are the permutations in which the relative position of the N gluons is $1_g, 2_g, \dots, N_g$, one gluon corresponding to the graviton $(N+1)_h$ can be inserted at any position between 1_g and l_g , the other gluon corresponding to the graviton can be inserted at any position between l_g and i_g (See Fig. 1(a) and (b)).

B. D_2 relations for tree amplitudes with two positive helicity gravitons, $N - 2$ positive helicity gluons and two negative helicity gluons

The tree amplitude (15) and (16), with $M = 2$ can be reduced to

$$\mathcal{A}(1_g^-, 2_g^+, \dots, i_g^-, \dots, N_g^+, (N+1)_h^+, (N+2)_h^+) = \mathbb{A} + \mathbb{B} + \mathbb{C}, \quad (22)$$

³ In this section, we let the gluons corresponding to a graviton take the momentum of the graviton for convenience. This is a little different from in the sections above where each gluon from a graviton take half of the momentum of the graviton. However, a redefinition of the momentum $k \rightarrow \frac{1}{2}k$ only contribute a constant factor which does not affect our discussions.

where \mathbb{A} , \mathbb{B} and \mathbb{C} are

$$\begin{aligned}
\mathbb{A} &= ig^{N-2} \left(-\frac{\kappa}{2}\right)^2 \frac{\langle 1i \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle N1 \rangle} \\
&\times \sum_{l \in \{g^+\}} s_{N+1, N+2} s_{l, N+1} \frac{\langle 1l \rangle}{\langle 1, N+1 \rangle \langle N+1, l \rangle} \frac{\langle li \rangle}{\langle l, N+1 \rangle \langle N+1, i \rangle} \\
&\times \frac{\langle 1, N+1 \rangle}{\langle 1, N+2 \rangle \langle N+2, N+1 \rangle} \frac{\langle N+1, i \rangle}{\langle N+1, N+2 \rangle \langle N+2, i \rangle},
\end{aligned} \tag{23}$$

$$\begin{aligned}
\mathbb{B} &= ig^{N-2} \left(-\frac{\kappa}{2}\right)^2 \frac{\langle 1i \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle N1 \rangle} \\
&\times \sum_{l \in \{g^+\}} s_{N+2, N+1} s_{l, N+2} \frac{\langle 1l \rangle}{\langle 1, N+2 \rangle \langle N+2, l \rangle} \frac{\langle li \rangle}{\langle l, N+2 \rangle \langle N+2, i \rangle} \\
&\times \frac{\langle 1, N+2 \rangle}{\langle 1, N+1 \rangle \langle N+1, N+2 \rangle} \frac{\langle N+2, i \rangle}{\langle N+2, N+1 \rangle \langle N+1, i \rangle},
\end{aligned} \tag{24}$$

$$\begin{aligned}
\mathbb{C} &= ig^{N-2} \left(-\frac{\kappa}{2}\right)^2 \frac{\langle 1i \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle N1 \rangle} \\
&\times \sum_{l \in \{g^+\}} s_{l, N+2} \frac{\langle l1 \rangle}{\langle N+2, 1 \rangle \langle l, N+2 \rangle} \frac{\langle li \rangle}{\langle l, N+2 \rangle \langle N+2, i \rangle} \\
&\times \sum_{k \in \{g^+\}} s_{k, N+1} \frac{\langle k1 \rangle}{\langle N+1, 1 \rangle \langle l, N+1 \rangle} \frac{\langle ki \rangle}{\langle k, N+1 \rangle \langle N+1, i \rangle}.
\end{aligned} \tag{25}$$

We first look at \mathbb{A} part. $\frac{\langle 1l \rangle}{\langle 1, N+1 \rangle \langle N+1, l \rangle}$ and $\frac{\langle li \rangle}{\langle l, N+1 \rangle \langle N+1, i \rangle}$ can split into sums of terms as in (18) and (19). For a given l , as in $M = 1$ case, they insert the two gluons corresponding to the graviton $(N+1)_h$ into positions between $1, l$ and l, i respectively. After this insertion, we consider the insertion of gluons corresponding to $(N+2)_h$. With the eikonal identity (A9), for $1 < l < i$, $\frac{\langle 1, N+1 \rangle}{\langle 1, N+2 \rangle \langle N+2, N+1 \rangle}$ and $\frac{\langle N+1, i \rangle}{\langle N+1, N+2 \rangle \langle N+2, i \rangle}$ in (23) can be given as

$$\begin{aligned}
\frac{\langle 1, N+1 \rangle}{\langle 1, N+2 \rangle \langle N+2, N+1 \rangle} &= \left(\frac{\langle 1, r \rangle}{\langle 1, N+2 \rangle \langle N+2, r \rangle} + \frac{\langle r, N+1 \rangle}{\langle r, N+2 \rangle \langle N+2, N+1 \rangle} \right), \\
\frac{\langle N+1, i \rangle}{\langle N+1, N+2 \rangle \langle N+2, i \rangle} &= \left(\frac{\langle t+1, i \rangle}{\langle t+1, N+2 \rangle \langle N+2, i \rangle} + \frac{\langle N+1, t+1 \rangle}{\langle N+1, N+2 \rangle \langle N+2, t+1 \rangle} \right),
\end{aligned} \tag{26}$$

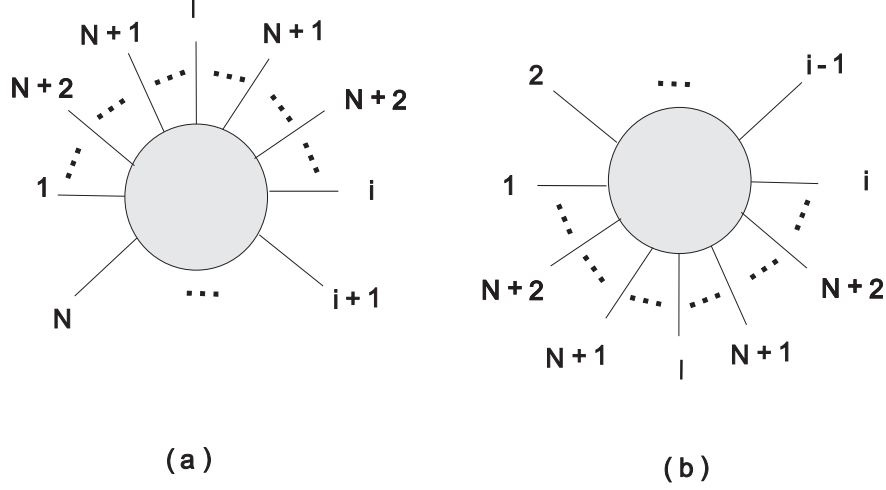


FIG. 2: Positions of the gluons corresponding to the two gravitons $(N+1)_h$ and $(N+2)_h$ for (a) $1 < l < i$, (b) $i < l \leq N$ in \mathbb{A} part.

while for $i < l \leq N$,

$$\begin{aligned} \frac{\langle i, N+1 \rangle}{\langle i, N+2 \rangle \langle N+2, N+1 \rangle} &= \left(\frac{\langle i, r \rangle}{\langle i, N+2 \rangle \langle N+2, r \rangle} + \frac{\langle r, N+1 \rangle}{\langle r, N+2 \rangle \langle N+2, N+1 \rangle} \right), \\ \frac{\langle N+1, 1 \rangle}{\langle N+1, N+2 \rangle \langle N+2, 1 \rangle} &= \left(\frac{\langle t+1, 1 \rangle}{\langle t+1, N+2 \rangle \langle N+2, 1 \rangle} + \frac{\langle N+1, t+1 \rangle}{\langle N+1, N+2 \rangle \langle N+2, t+1 \rangle} \right). \end{aligned} \quad (27)$$

The first term of the sum in each line of (26) and (27) can split into sum over adjacent points again, for example, the first term in the first line of (26) can be expressed as

$$\sum_{p=1}^{r-1} \frac{\langle p, p+1 \rangle}{\langle p, N+2 \rangle \langle N+2, p+1 \rangle}. \quad (28)$$

For a given r , we have inserted a gluon corresponding to $(N+1)_h$ between r and $r+1$, then (28) insert a gluon corresponding to $(N+2)_h$ at a position between 1 and r . The second term in the first line of (26) insert an gluon corresponding to $(N+2)_h$ between r and the gluon corresponding to $(N+1)_h$. Thus the first line of (26) just insert a gluon corresponding to $(N+2)_h$ at the positions between 1 and the gluon corresponding to $(N+1)_h$, where the gluon corresponding to $(N+1)_h$ have been inserted at positions between 1 and l . In a same

way, the second line of (26) insert the other gluon corresponding to $(N + 2)_h$ at positions between the gluon corresponding to $(N + 1)_h$ and i , where this gluon corresponding to $(N + 1)_h$ have been inserted at a position between l and i (See Fig. 2 (a)). Following a similar discussion, for the case of $i < l \leq N$, we insert the two gluons corresponding to $(N + 1)_h$ at positions between i, l and $l, 1$ respectively. Then insert one gluon corresponding to $(N + 2)_h$ at the positions between i and the gluon corresponding to $(N + 1)_h$ which is between i, l . Insert the other gluon corresponding to $(N + 2)_h$ at the positions between the other gluon corresponding to $(N + 1)_h$ and 1(See Fig. 2 (b)). The \mathbb{A} part then satisfy the relation

$$\mathbb{A} = \sum_{l \in \{g^+\}} s_{l,N+1} s_{N+1,N+2} \sum_{P_1} \mathcal{A}_{MHV}^{N+4}(P_1), \quad (29)$$

where for a given l , P_1 are the possible insertions of the gluons corresponding to the two gravitons. These insertions has the form $1, \dots, N + 2, \dots, N + 1, \dots, l, \dots, N + 1, \dots, N + 2, \dots, i, \dots, N$ for $1 < l < i$ and $1, \dots, i, \dots, N + 2, \dots, N + 1, \dots, l, \dots, N + 1, \dots, N + 2, \dots$ for $i < l \leq N$.

The \mathbb{B} part can be derived from \mathbb{A} part by the replacement $N + 1 \leftrightarrow N + 2$ and is given as

$$\mathbb{B} = \sum_{l \in \{g^+\}} s_{l,N+2} s_{N+2,N+1} \sum_{P_2} \mathcal{A}_{MHV}^{N+4}(P_2), \quad (30)$$

where for a given l , P_2 are the possible insertions of the gluons corresponding to the two gravitons. These insertions has the form $1, \dots, N + 1, \dots, N + 2, \dots, l, \dots, N + 2, \dots, N + 1, \dots, i, \dots, N$ for $1 < l < i$ (See Fig. 3 (a)) and $1, \dots, i, \dots, N + 1, \dots, N + 2, \dots, l, \dots, N + 2, \dots, N + 1, \dots$ for $i < l \leq N$ (See Fig. 3 (b)).

Now we consider the \mathbb{C} part. Both the second and the third lines in (25), can split into the forms of (18) and (19). Then for a given l , each term in the second line of the expression (25) insert the two gluons corresponding to $(N + 2)_h$ at the positions between $1, l$ and l, i

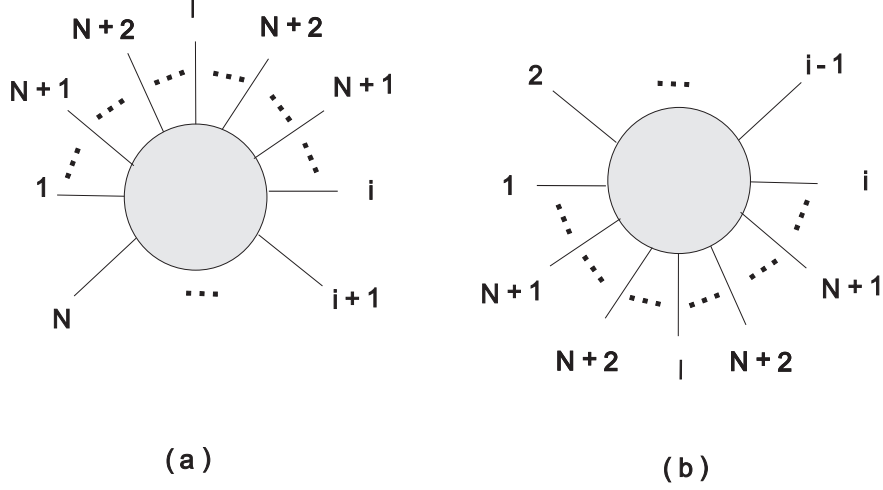


FIG. 3: Positions of the gluons corresponding to the two gravitons $(N+1)_h$ and $(N+2)_h$ for (a) $1 < l < i$, (b) $i < l \leq N$ in \mathbb{B} part.

respectively. Then for any given k , each term in the second line of (25) insert the two gluons corresponding to $(N+1)_h$ at the positions between 1, k and k , i respectively. The \mathbb{C} part then becomes

$$\mathbb{C} = \sum_{l,k \in g^+} s_{l,N+2} s_{k,N+1} \sum_{P_3} \mathcal{A}_{MHV}^{N+4}(P_3). \quad (31)$$

where P_3 are the insertions of the four gluons corresponding to the two gravitons $(N+2)_h$ and $(N+1)_h$. In these insertions, two gluons corresponding to $(N+2)_h$ are inserted at the positions between 1, l and l , i respectively, then two gluons corresponding to $(N+1)_h$ are inserted at the positions between 1, k and k , i respectively(See Fig. 4).

After considering all the contributions from \mathbb{A} , \mathbb{B} and \mathbb{C} , we give the D_2 relation

$$\begin{aligned} & \mathcal{A}(1_g^-, 2_g^+, \dots, i_g^-, \dots, N_g^+, (N+1)_h^+, (N+2)_h^+) \\ &= \sum_{l \in \{g^+\}} s_{l,N+1} s_{N+1,N+2} \sum_{P_1} \mathcal{A}_{MHV}^{N+4}(P_1) + \sum_{l \in \{g^+\}} s_{l,N+2} s_{N+2,N+1} \sum_{P_2} \mathcal{A}_{MHV}^{N+4}(P_2) \\ &+ \sum_{l,k \in g^+} s_{l,N+2} s_{k,N+1} \sum_{P_3} \mathcal{A}_{MHV}^{N+4}(P_3). \end{aligned} \quad (32)$$

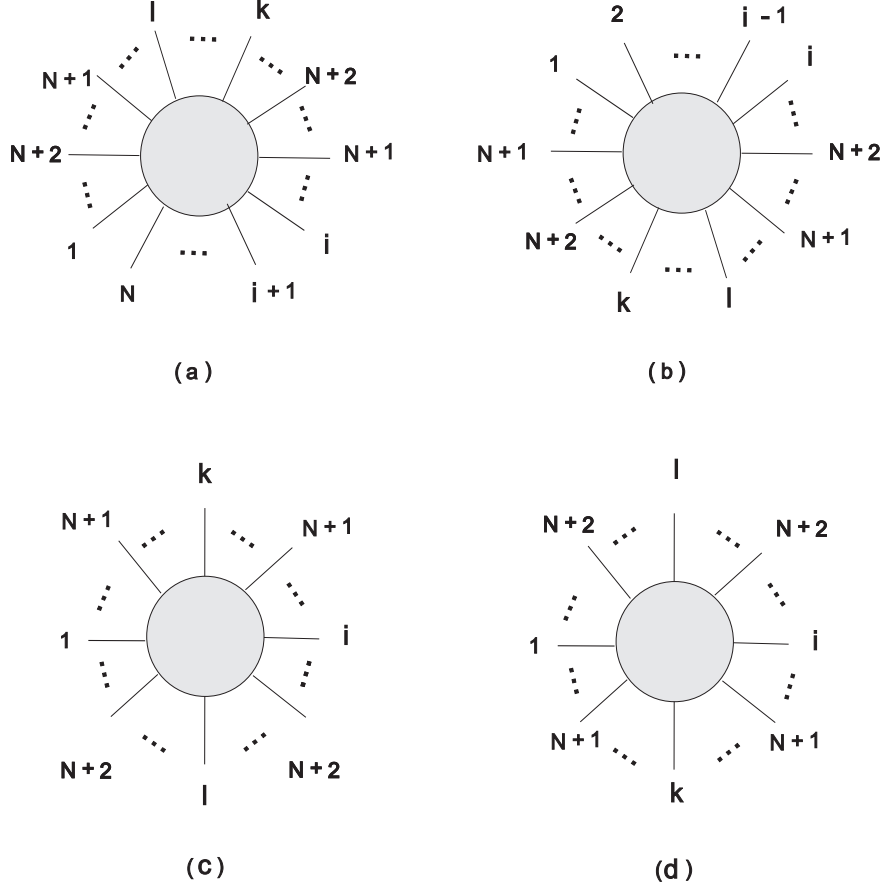


FIG. 4: Positions of the gluons corresponding to the two gravitons $(N+1)_h$ and $(N+2)_h$ for (a) $1 < l < i$, $1 < k < i$, (b) $i < l \leq N$, $i < k \leq N$ in (c) $i < l \leq N$, $1 < k < i$ and (d) $1 < l < i$, $i < k \leq N$ in \mathbb{C} part. Here we first insert the two gluons corresponding to $(N+2)_h$ between 1, l and l , i respectively. We then insert two gluons corresponding to $(N+1)_h$ between 1, k and k , i respectively.

C. D_2 relations for arbitrary tree amplitudes with N gluons minimally coupled to M gravitons

The amplitudes with more gravitons are more complicated. However the discussions are similar with those for amplitudes with one and two gravitons. The amplitude with N gluons minimally coupled to M gravitons, where two gluons take negative helicity and other legs

take positive helicity (15) and (16) can be given by a sum of terms. Each term has the form

$$\begin{aligned}
& ig^{N-2} \left(-\frac{\kappa}{2}\right)^M \frac{\langle 1i \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle N1 \rangle} \\
& \times \frac{\langle l_1 1 \rangle \langle l_1 i \rangle [l_1 n_1^1]}{\langle n_1^1 1 \rangle \langle n_1^1 i \rangle \langle n_1^1 n_2^1 \rangle} \times \frac{\langle n_1^1 1 \rangle \langle n_1^1 i \rangle [n_1^1 n_2^1]}{\langle n_2^1 1 \rangle \langle n_2^1 i \rangle \langle n_1^1 n_2^1 \rangle} \times \dots \\
& \times \frac{\langle l_2 1 \rangle \langle l_2 i \rangle [l_2 n_1^2]}{\langle n_1^2 1 \rangle \langle n_1^2 i \rangle \langle n_1^2 n_2^2 \rangle} \times \frac{\langle n_1^2 1 \rangle \langle n_1^2 i \rangle [n_1^2 n_2^2]}{\langle n_2^2 1 \rangle \langle n_2^2 i \rangle \langle n_1^2 n_2^2 \rangle} \times \dots \\
& \vdots \\
& \times \frac{\langle l_N 1 \rangle \langle l_N i \rangle [l_N n_1^N]}{\langle n_1^N 1 \rangle \langle n_1^N i \rangle \langle n_1^N n_2^N \rangle} \times \frac{\langle n_1^N 1 \rangle \langle n_1^N i \rangle [n_1^N n_2^N]}{\langle n_2^N 1 \rangle \langle n_2^N i \rangle \langle n_1^N n_2^N \rangle} \times \dots,
\end{aligned} \tag{33}$$

where each l_i can be any positive helicity gluon, and $n_1^1, n_2^1, \dots, n_1^2, n_2^2, \dots, \dots, n_1^N, n_2^N, \dots$ is a permutation of all the gravitons. Following the discussions on amplitudes with one and two gravitons, this term can be given by sum of MHV amplitudes with $N + 2M$ gluons with appropriate factors. The first line insert two gluons corresponding to n_1^1 between 1, l_1 and l_1, i respectively, then insert two gluons corresponding to n_2^1 between 1, one gluon corresponding to n_1^1 and n_1^1 , the other gluon corresponding to n_1^1 respectively, ... After inserting all the gluons corresponding to gravitons in the first line, we insert two gluons corresponding to n_1^2 between 1, l_2 and l_2, i respectively, then insert two gluons corresponding to n_2^2 between 1, one gluon corresponding to n_1^2 and n_1^2 , the other gluon corresponding to n_1^2 respectively, ... In this way, we insert all the $2M$ gluons corresponding to the M gravitons into the amplitudes. The phase factor is $s_{l_1, n_1^1} s_{n_1^1, n_2^1} \dots s_{l_2, n_1^2} s_{n_1^2, n_2^2} \dots s_{l_N, n_1^N} s_{l_N, n_2^N} \dots$. At last, the amplitudes become MHV tree amplitudes with $N + 2M$ gluons, where the two gluons corresponding to a same graviton take the same momentum. This is just the D_2 relations in field theory.

In string theory, the D_2 relations are independent of helicity configurations of the legs. Then we expect the D_2 relations should have helicity-independent form. For example, in the relation for amplitudes with one graviton and N gluons given in Subsection IV A, we only sum over l corresponding to the gluons with positive helicity, and for each l , the two gluons

are inserted at the positions between 1, l and l , i respectively, the relation (21) depends on the relative positions of the two negative helicity gluons. Then we expect the relation (21) can be extended to that independent of the relative positions of the two negative helicity legs. In fact, in the expression (17), we can sum over l ($1 < l \leq N$). This is because $\langle li \rangle$ vanishes for $l = i$. Using the eikonal identity (A9) and the identity (A10) implied by momentum conservation, the amplitude becomes

$$\begin{aligned}
& \mathcal{A}(1_g^-, 2_g^+, \dots, i_g^-, \dots, N_g^+, (N+1)_h^+) \\
&= i g^{N-2} \left(\frac{\kappa}{2} \right) \sum_{1 < l \leq N} \frac{\langle 1i \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle N1 \rangle} \langle l, N+1 \rangle [l, N+1] \\
&\times \frac{\langle 1l \rangle}{\langle 1, N+1 \rangle \langle N+1, l \rangle} \frac{\langle l1 \rangle}{\langle l, N+1 \rangle \langle N+1, 1 \rangle}.
\end{aligned} \tag{34}$$

Then we repeat the discussions above. The amplitude satisfy the relation

$$\begin{aligned}
& \mathcal{A}(1_g^-, 2_g^+, \dots, i_g^-, \dots, N_g^+, (N+1)_h^+) \\
&= i g^{N-2} \left(\frac{\kappa}{2} \right) \sum_{1 < l \leq N} s_{l, N+1} \sum_{P'} \mathcal{A}_{MHV}^{N+2}(P'),
\end{aligned} \tag{35}$$

where we sum over all the external gluons. For any given l , P' in this relation denote all the permutations where one of the two gluons corresponding to the graviton is inserted at positions to the left of l and right of 1, the other one is inserted to the left of 1 and right of l (See Fig. 1(c)). Since this expression of the amplitude does not depend on the helicity configuration of the legs. We suggest that for any helicity configuration, the tree amplitude with N gluons minimally coupled to one graviton satisfy the relation

$$\begin{aligned}
& \mathcal{A}(1_g, 2_g, \dots, N_g, (N+1)_h) \\
&= i g^{N-2} \left(\frac{\kappa}{2} \right) \sum_{1 < l \leq N} s_{l, N+1} \sum_{P'} \mathcal{A}^{N+2}(P').
\end{aligned} \tag{36}$$

Though this extension will be more complicated, we expect there must be such extensions to the relations for arbitrary helicity configurations.

The D_2 relation for a given amplitude may have different expressions as in string theory. A tree amplitude for gauge-gravity minimal coupling can be expressed by different sets of pure-gluon partial tree amplitudes. The permutations of the legs and the factors in different expressions are different. However, the partial tree amplitudes of gluons are not independent of each other, there are relations among the partial tree amplitudes with gluons[6, 30]. Then the different expressions of the D_2 relation for a given amplitude can be related by the relations among pure-gluon partial tree amplitudes. To see this, we take amplitude with three gluons and one graviton as an example. In Section II, the relation is given by (6b), the factor is in s_{13} -channel. However, in Section IV, the relation is given by (36). For $N = 3$, we have

$$\begin{aligned} \mathcal{A}(1_g, 2_g, 3_g, 4_h) = g \left(-\frac{\kappa}{2} \right) & \left[s_{24} \mathcal{A}(1_g, 5_g, 2_g, 4_g, 3_g) + s_{24} \mathcal{A}(1_g, 4_g, 2_g, 3_g, 5_g) \right. \\ & \left. + s_{34} \mathcal{A}(1_g, 4_g, 2_g, 3_g, 5_g) + s_{34} \mathcal{A}(1_g, 2_g, 4_g, 3_g, 5_g) \right], \end{aligned} \quad (37)$$

where we denote the two gluons corresponding to 4_h by 4_g and 5_g . Then the amplitude is given by different expressions. In the last expression, $s_{2,4} = s_{1,3}$, $s_{24} + s_{34} = -s_{14} = -s_{23}$ and $s_{34} = s_{12}$. The second and the third terms can be given by one term with the factor $-s_{23}$. Using the relations among partial amplitudes, we have

$$\begin{aligned} \mathcal{A}(1_g, 4_g, 2_g, 3_g, 5_g) &= \frac{s_{13}}{s_{23}} \mathcal{A}(1_g, 5_g, 2_g, 4_g, 3_g), \\ \mathcal{A}(1_g, 2_g, 4_g, 3_g, 5_g) &= \frac{s_{13}}{s_{12}} \mathcal{A}(1_g, 5_g, 2_g, 4_g, 3_g). \end{aligned} \quad (38)$$

Then the two relations (37) and (6b) are equivalent.

V. CONCLUSION

In this paper, we study the amplitudes where gluons are minimally coupled with gravitons. We find the three- and four-point amplitudes satisfy the field theory limits of D_2 relations

in string theory. The left- and right-moving sectors are connected into a single one.

We give particular forms of the relations for the amplitude $\mathcal{A}(1_g^-, 2_g^+, \dots, i_g^-, \dots, N_g^+, (N+1)_h^+)$, and $\mathcal{A}(1_g^-, 2_g^+, \dots, i_g^-, \dots, N_g^+, (N+1)_h^+, (N+2)_h^+)$. We extend the relation to arbitrary helicity configurations for $N+1$ case. The discussions can be extended to arbitrary legs with arbitrary helicity configurations. The tree amplitude with N gluons and M gravitons can be expressed by sum of amplitudes for $N+2M$ gluons with appropriate factors. The relation for a given amplitude is not unique, because there are relations among pure-gluon partial amplitudes.

Though the D_2 relations and KLT factorization relations only hold on D_2 and S_2 respectively in string theory, the field theory limits of both two relations hold in minimal coupling theory of gauge and gravity. This is because we have two different methods to incorporate gauge degree of freedom in string theory.

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Appendix A: Spinor helicity formalism

Here we given the useful properties of spinor helicity formalism[24–26] Positive and negative helicity spinor

$$|i^\pm\rangle \equiv |k_i^\pm\rangle \equiv u_\pm(k_i) = v_\mp(k_i), \langle i^\pm| \equiv \langle k_i^\pm| \equiv \bar{u}_\pm(k_i) = \bar{v}_\pm(k_i), \quad (\text{A1})$$

where u and v are positive and negative energy solutions of Dirac equation.

$$\begin{aligned}\langle ij \rangle &\equiv \langle i^- | j^+ \rangle = \sqrt{|s_{ij}|} e^{i\phi_{ij}}, \\ [ij] &\equiv \langle i^+ | j^- \rangle = \sqrt{|s_{ij}|} e^{-i(\phi_{ij} + \pi)}.\end{aligned}\tag{A2}$$

Momentum

$$\langle i^\pm | \gamma^\mu | i^\pm \rangle = 2k_i.\tag{A3}$$

Polarization vector

$$\epsilon_\mu^\pm(k, q) = \pm \frac{\langle q^\pm | \gamma_\mu | k^\mp \rangle}{\sqrt{2} \langle q^\mp | k^\pm \rangle},\tag{A4}$$

where q is reference momentum, reflecting the freedom of on-shell gauge transformation, k is the vector boson momentum.

Useful properties:

$$\langle ij \rangle [ji] = s_{ij}\tag{A5}$$

antisymmetry

$$\langle ij \rangle = -\langle ji \rangle, [ij] = -[ji], \langle ii \rangle = [ii] = 0,\tag{A6}$$

Fierz rearrangement

$$\langle i^+ | \gamma^\mu | j^+ \rangle \langle k^+ | \gamma_\mu | l^+ \rangle = 2[ik] \langle lj \rangle,\tag{A7}$$

charge conjugation

$$\langle i^+ | \gamma^\mu | j^+ \rangle = \langle j^- | \gamma^\mu | i^- \rangle,\tag{A8}$$

eikonal identity

$$\sum_{i=j}^{k-1} \frac{\langle i, i+1 \rangle}{\langle iq \rangle \langle q, i+1 \rangle} = \frac{\langle jk \rangle}{\langle jq \rangle \langle qk \rangle}.\tag{A9}$$

in an N -point amplitude, momentum conservation imply

$$\sum_{i=1, i \neq k}^n [ji] \langle ik \rangle = 0.\tag{A10}$$

The amplitudes with all positive helicity gluons are zero. The amplitudes for gluons with one negative helicity and others positive helicity are zero. The amplitude for N gluons with two negative and $N - 2$ positive(MHV) helicities can be given[6]

$$\mathcal{A}(1^+, \dots, i^-, \dots, j^-, \dots, n^+) = i \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle}. \quad (\text{A11})$$

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